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Second-order Lagrangians admitting a second-order Hamilton–Cartan formalism

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Abstract. The Poincaré–Cartan (PC) form of a Lagrangian on the bundle $J^2 = J^2(N, M)$ is, as a general rule, defined on J^3 thus leading to a non-equivalence between Euler–Lagrange and Hamilton–Cartan equations. This naturally leads to the problem of determining what Lagrangians have a PC form projectable onto J^2, as they will then admit a second-order Hamiltonian formalism. There are specific examples of this phenomenon in field theory. This paper provides an explicit classification of such Lagrangians.

1. Introduction

As is well known, the extremals of the functional defined by a first-order Lagrangian density $L\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ on $J^1 = J^1(N, M)$, where M, N are C^∞ manifolds of dimensions $\dim N = n$, $\dim M = m$, can be viewed as the solutions to the so-called Hamilton–Cartan equation; i.e., a field $f : N \to M$ is an extremal if and only if $(j^1 f)^* i_X \,\mathrm{d}\Theta = 0$ for all $X \in TJ^1$ (cf [1, formula (3.7)]), where

$$\Theta = H\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n + \sum_{i=1}^{m}\sum_{j=1}^{n}(-1)^{j-1}\frac{\partial L}{\partial y^i_{(j)}}\,\mathrm{d}y^i \wedge \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^j} \wedge \cdots \wedge \mathrm{d}x^n \tag{1}$$

is the Poincaré–Cartan form (hereafter PC form) attached to L, the function H is given by

$$H = L - \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\partial L}{\partial y^i_{(j)}}\,y^i_{(j)} \tag{2}$$

and $(y^i)_{1\leqslant i\leqslant m}$, $(x^j)_{1\leqslant j\leqslant n}$, (x^j, y^i, y^i_j) are coordinates on M, N, and the induced coordinate system on the one-jet bundle, respectively.

The basic point in the Hamilton–Cartan formalism is that any solution $\bar{f} : N \to J^1$ to the Hamilton–Cartan equation is automatically holonomic whenever the Lagrangian is regular. Therefore, the extremals of a regular variational problem are the n-dimensional solutions of an exterior differential system on J^1.

However, for a second-order Lagrangian density, the PC form lies on J^3 and the corresponding Hamilton–Cartan equation should now read

$$\bar{f}^* i_X \,\mathrm{d}\Theta = 0 \qquad \forall X \in TJ^3$$

where $\bar{f} : N \to J^3$ is a section of the natural projection $J^3 \to N$. There are three cases as follows:

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(1) For $n = 1$ (i.e. in second-order mechanics) the theory is the same as in the first order (see [2]).

(2) Assume $n \geqslant 2$; then,

(2.1) the set of solutions to the Hamilton–Cartan equations is larger than that of Euler–Lagrange equations for *every* Lagrangian, and

(2.2) with a suitable notion of regularity (stronger than the classical one), it is proved in [3] that any solution $\bar{f} : N \to J^3$ to the Hamilton–Cartan equation is holonomic up to the second order.

As we can never reach the third-order holonomy, it is natural to pose the question of determining the second-order Lagrangians whose PC form is projectable onto J^2.

In field theory there are two basic examples of Lagrangians, the PC form of which projects onto a jet bundle of lower order. The first one is the Lagrangian L_1 that governs the interaction of the Dirac electron field with an electromagnetic potential. In this case, L_1 is a first-order Lagrangian whose PC form projects onto J^0 (for the details see, e.g., [4, section 7.2]). Because of this, Dirac's equation is of first order. Actually, L_1 is a first-degree polynomial in the derivatives y^i_j. The second example is the Lagrangian L_2 defined by the scalar curvature on the two-jet bundle of Lorentzian metrics on space–time. Now, the PC form of L_2 not only projects onto J^2 but also onto J^1, thus leading one to a first-order Hamiltonian formalism.

The goal of this paper is to provide the classification of second-order Lagrangians, the PC form of which projects onto J^2 and therefore admit a true second-order Hamiltonian formalism. We should remark that in the above examples the PC form projects onto a lower-order jet bundle due to the fact that the Lagrangian is a first-degree polynomial on the higher derivatives. The converse only holds for $n = 1$ but it is no longer true for any dimension $n \geqslant 2$. In fact, as we shall prove later on, the first-degree Lagrangians in the second derivatives are a small subset of the Lagrangians, the PC form of which projects onto the second jet bundle.

2. Preliminaries

2.1. Jet bundle notations

We use multi-index notation. A multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ of length n is an element $\alpha \in \mathbb{N}^n$, where $|\alpha| = \alpha_1 + \cdots + \alpha_n$ is the order of α. The sum of two multi-indices is defined componentwise. Multi-indices are ordered as follows: $\alpha \leqslant \beta$ means $\alpha_i \leqslant \beta_i$ for all $i = 1, \ldots, n$. Similarly, we set

$$\alpha! = \alpha_1! \ldots \alpha_n! \qquad \binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha - \beta)!} \qquad \beta \leqslant \alpha.$$

For every $1 \leqslant j \leqslant n$ we denote by (j) the multi-index α whose entries are given by $\alpha_k = \delta_{jk}$, $1 \leqslant k \leqslant n$. We also set $(jk) = (j) + (k)$, $(jkl) = (j) + (k) + (l)$, etc. The Kronecker symbol of multi-indices is defined as usual; that is, $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$, $\delta_{\alpha\beta} = 0$ otherwise. Below we will need the following lemma whose proof is straightforward.

Lemma 2.1. *With the above hypotheses and assumptions we have*

$$\delta_{(hi)(jk)} = \delta_{hj}\delta_{ik} + \delta_{hk}\delta_{ij} - \delta_{hi}\delta_{jk}\delta_{hk}.$$

Let (x^j, y^h, y^i_α), $h, i = 1, \ldots, m$, $1 \leqslant j \leqslant n$, $0 \leqslant |\alpha| \leqslant r$ be the coordinates induced on J^r, $0 \leqslant r \leqslant 3$, by the systems $(y^i)_{1 \leqslant i \leqslant m}$, $(x^j)_{1 \leqslant j \leqslant n}$ on M, N, respectively, with $y^i_0 = y^i$; i.e. $y^i_\alpha(j^r_x f) = (\partial^{|\alpha|}(y^i \circ f)/\partial(x^1)^{\alpha_1} \ldots \partial(x^n)^{\alpha_n})(x)$.

2.2. Second-order PC form

We recall that a PC form naturally associated to a second-order Lagrangian is locally given by (cf [5])

$$\Theta = H\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n + \sum_{i=1}^{m}\sum_{j=1}^{n} L_{ij}\,\mathrm{d}y^i\wedge\mathrm{d}x^1\wedge\cdots\wedge\widehat{\mathrm{d}x^j}\wedge\cdots\wedge\mathrm{d}x^n$$
$$+\sum_{i=1}^{m}\sum_{j,k=1}^{n} L^i_{jk}\,\mathrm{d}y^i_{(k)}\wedge\mathrm{d}x^1\wedge\cdots\wedge\widehat{\mathrm{d}x^j}\wedge\cdots\wedge\mathrm{d}x^n \tag{3}$$

where

$$\begin{aligned} L_{ij} &= (-1)^{j-1}\frac{\partial L}{\partial y^i_{(j)}} + \sum_{k=1}^{n}\frac{(-1)^j}{2-\delta_{jk}}\frac{\mathrm{d}}{\mathrm{d}x^k}\left(\frac{\partial L}{\partial y^i_{(jk)}}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x^k} &= \frac{\partial}{\partial x^k} + \sum_{i=1}^{m}\sum_{|\alpha|=0}^{\infty} y^i_{\alpha+(k)}\frac{\partial}{\partial y^i_\alpha}\end{aligned} \tag{4}$$

is the total derivative operator corresponding to $\partial/\partial x^k$ on jet bundles, and

$$\begin{aligned} H &= L + \sum_{i=1}^{m}\sum_{j=1}^{n}(-1)^j L_{ij} y^i_{(j)} - \sum_{i=1}^{m}\sum_{j,k=1}^{n}\frac{1}{2-\delta_{jk}}\frac{\partial L}{\partial y^i_{(jk)}} y^i_{(jk)}\\ L^i_{jk} &= \frac{(-1)^{j-1}}{2-\delta_{jk}}\frac{\partial L}{\partial y^i_{(jk)}}.\end{aligned} \tag{5}$$

3. Statement of the result

Before stating the theorem, we first introduce some notations. For every $s = 1,\dots,n$, let $\mathcal{I}_s$ be the set of increasing indices $I = (i_1,\dots,i_s)$, where $i_1,\dots,i_s$ are integers such that $1 \leqslant i_1 < \cdots < i_s \leqslant n$. For $s = 0$ we set $\mathcal{I}_0 = \emptyset$. The elements $I \in \mathcal{I}_s$ are in one-to-one correspondence with the increasing maps $I : \{1,\dots,s\} \to \{1,\dots,n\}$, that is, $I(1) < \cdots < I(s)$. We order $\mathcal{I}_s$ lexicographically; that is, $I < J$ means there exists an index $h = 0,\dots,s-1$ such that $i_1 = j_1,\dots,i_h = j_h, i_{h+1} < j_{h+1}$. Bearing these notations in mind, we set for every $h, i = 1,\dots,m$; $I \leqslant J$, $I, J \in \mathcal{I}_s$; $2 \leqslant s \leqslant n$,

$$\Delta^{hi}_{IJ} = \begin{vmatrix} y^h_{(i_1j_1)} & y^i_{(i_1j_2)} & \cdots & y^i_{(i_1j_s)}\\ y^h_{(i_2j_1)} & y^i_{(i_2j_2)} & \cdots & y^i_{(i_2j_s)}\\ \vdots & \vdots & \ddots & \vdots\\ y^h_{(i_sj_1)} & y^i_{(i_sj_2)} & \cdots & y^i_{(i_sj_s)}\end{vmatrix} + \begin{vmatrix} y^i_{(i_1j_1)} & y^h_{(i_1j_2)} & \cdots & y^i_{(i_1j_s)}\\ y^i_{(i_2j_1)} & y^h_{(i_2j_2)} & \cdots & y^i_{(i_2j_s)}\\ \vdots & \vdots & \ddots & \vdots\\ y^i_{(i_sj_1)} & y^h_{(i_sj_2)} & \cdots & y^i_{(i_sj_s)}\end{vmatrix}$$
$$+\cdots+\begin{vmatrix} y^i_{(i_1j_1)} & y^i_{(i_1j_2)} & \cdots & y^h_{(i_1j_s)}\\ y^i_{(i_2j_1)} & y^i_{(i_2j_2)} & \cdots & y^h_{(i_2j_s)}\\ \vdots & \vdots & \ddots & \vdots\\ y^i_{(i_sj_1)} & y^i_{(i_sj_2)} & \cdots & y^h_{(i_sj_s)}\end{vmatrix}. \tag{6}$$

We also set:
(i) $\Delta^{hi}_{\emptyset\emptyset} = 1$ for $s = 0$.
(ii) $\Delta^{hi}_{i_1j_1} = y^h_{(i_1j_1)}$ for $s = 1$.
(iii) $\Delta_{IJ} = \Delta_{JI}$ whenever $I > J$.

(iv) Given an index $1 \leqslant k \leqslant n$ and an element $I \in \mathcal{I}_s$, we define

$$\Delta^{hi}_{I-k,J} = \begin{cases} \Delta^{hi}_{I-\{k\},J} & \text{if} \quad k \in I \\ 0 & \text{if} \quad k \notin I \end{cases}$$

where $I - \{k\}$ denotes the increasing sequence obtained by deleting the index k in I; that is,

$$I - k = (i_1, \ldots, \hat{i}_t, \ldots, i_s) \qquad i_t = k$$

and similarly for $\Delta^{hi}_{I,J-k}$.

Remark 3.1. We have

(1) $\Delta^{hi}_{IJ} = \Delta^{ih}_{IJ}$ for $s = 2$.
(2) $\Delta^{hi}_{I-k-l,J} = \Delta^{hi}_{I-\{k,l\},J}$, if $k \neq l$, but $\Delta^{hi}_{I-k-k,J} \neq \Delta^{hi}_{I-\{k\},J}$, as the left-hand side is always zero whereas the right-hand side does not vanish necessarily.
(3) The functions Δ^{hi}_{IJ} are not linearly independent; for example, we have

$$\Delta^{hi}_{(12a_3\cdots a_s)(34a_3\cdots a_s)} - \Delta^{hi}_{(13a_3\cdots a_s)(24a_3\cdots a_s)} + \Delta^{hi}_{(14a_3\cdots a_s)(23a_3\cdots a_s)} = 0.$$

Theorem 3.2. *The PC form of a second-order Lagrangian L is projectable onto $J^2(N, M)$ if and only if L can be written as*

$$L = \sum_{s=0}^{n} \sum_{I \leqslant J, I, J \in \mathcal{I}_s} \sum_{h,i=1}^{m} f^{hi}_{IJ} \Delta^{hi}_{IJ} \tag{7}$$

for certain differentiable functions f^{hi}_{IJ} on $J^1(N, M)$.

4. The associated linear PDE system

Proposition 4.1. *The PC form of a second-order Lagrangian L is projectable onto J^2 if and only if the following system of linear partial differential equations (PDEs) holds:*

$$\sum_{j=1}^{n} \frac{1}{2 - \delta_{jk}} \frac{\partial^2 L}{\partial y^h_{\alpha-(j)} \partial y^i_{(jk)}} = 0 \tag{8}$$

for every system of indices $\alpha \in \mathbb{N}^n$, $|\alpha| = 3$; $1 \leqslant k \leqslant n$; $h, i = 1, \ldots, m$.

Proof. A necessary condition for Θ to be projectable onto J^2 is that L_{ij} depends only on up to the second derivatives. Hence, from the formula (4), it follows that (8) must hold for every multi-index $|\alpha| = 3$ and every $h, i = 1, \ldots, m$, $1 \leqslant k \leqslant n$. This condition is also sufficient as, if equations (8) are satisfied, then $H \in C^\infty(J^2)$, as it follows from formula (5). Consequently, the Lagrangians, the PC form of which is projectable onto J^2, are the solutions of a system of homogeneous linear partial differential equations with constant coefficients which are to be determined explicitly. □

There are three types of equations in (8).

I. If $\alpha = (aaa)$, $1 \leqslant a \leqslant n$, then (8) becomes

$$\frac{\partial^2 L}{\partial y^h_{(aa)} \partial y^i_{(ak)}} = 0$$

with $h, i = 1, \ldots, m$; $a, k = 1, \ldots, n$.

II.1. If $\alpha = (aab)$, $a \neq b \neq k \neq a$, then for $a, b, k = 1, \ldots, n$, equation (8) reads:

(a) for $h \neq i$,

$$\frac{\partial^2 L}{\partial y^h_{(ab)} \partial y^i_{(ak)}} + \frac{\partial^2 L}{\partial y^h_{(aa)} \partial y^i_{(bk)}} = 0$$

with $h, i = 1, \ldots, m$, and

(b) for $h = i$,

$$\frac{\partial^2 L}{\partial y^h_{(ab)} \partial y^h_{(ak)}} + \frac{\partial^2 L}{\partial y^h_{(aa)} \partial y^h_{(bk)}} = 0$$

with $h = 1, \ldots, m$; $b < k$.

II.2. We assume $\alpha = (aab)$, $k \in \{a, b\}$, $a, b, k = 1, \ldots, n$. If $k = a$, then equation (8) reads

$$2\frac{\partial^2 L}{\partial y^h_{(ab)} \partial y^i_{(aa)}} + \frac{\partial^2 L}{\partial y^h_{(aa)} \partial y^i_{(ab)}} = 0$$

and each term vanishes because of the equation in (I). Hence we have the case $k = b$ remaining and we further suppose $a < b$ by index symmetry. Then:

(a) for $h \neq i$,

$$\frac{\partial^2 L}{\partial y^h_{(ab)} \partial y^i_{(ab)}} + 2\frac{\partial^2 L}{\partial y^h_{(aa)} \partial y^i_{(bb)}} = 0$$

with $h, i = 1, \ldots, m$, and

(b) for $h = i$,

$$\frac{\partial^2 L}{\partial y^h_{(ab)} \partial y^h_{(ab)}} + 2\frac{\partial^2 L}{\partial y^h_{(aa)} \partial y^h_{(bb)}} = 0$$

with $h = 1, \ldots, m$.

III.1. If $\alpha = (abc)$, with $a < b < c$, $k \neq a, b, c$, for $h \neq i$ we have

$$\frac{\partial^2 L}{\partial y^h_{(bc)} \partial y^i_{(ak)}} + \frac{\partial^2 L}{\partial y^h_{(ac)} \partial y^i_{(bk)}} + \frac{\partial^2 L}{\partial y^h_{(ab)} \partial y^i_{(ck)}} = 0$$

with $h, i = 1, \ldots, m$; $a, b, c, k = 1, \ldots, n$, and

III.2. if $\alpha = (abc)$, with $a < b < c < k$, for $h = i$,

$$\frac{\partial^2 L}{\partial y^h_{(bc)} \partial y^h_{(ak)}} + \frac{\partial^2 L}{\partial y^h_{(ac)} \partial y^h_{(bk)}} + \frac{\partial^2 L}{\partial y^h_{(ab)} \partial y^h_{(ck)}} = 0$$

with $h = 1, \ldots, m$.

In the case (III.1) it is proved that half of the equations depend linearly on the other half, and in the remaining cases the equations are linearly independent. Hence, the number of essential equations in (8) is

$$\begin{aligned}\mu = \mu(m, n) &= \tfrac{1}{2}(m+1)mn + m^2 n(n-1) + m(m-1)n(n-1)(n-2) \\ &\quad + \tfrac{1}{2}mn(n-1)(n-2) + \tfrac{1}{2}m(m-1)n(n-1) + \tfrac{1}{2}mn(n-1) \\ &\quad + \tfrac{1}{12}m(m-1)n(n-1)(n-2)(n-3) + \tfrac{1}{24}mn(n-1)(n-2)(n-3) \\ &= \frac{m(2m-1)}{24}n^4 + \frac{m(2m-1)}{4}n^3 - \frac{m(14m-25)}{24}n^2 + \frac{m(2m-1)}{4}n. \end{aligned} \tag{9}$$

5. Proof of theorem 3.2

5.1. The 'if' part

First, we check that the functions Δ^{hi}_{IJ} introduced in section 3 satisfy the system (8). As it is linear, all functions in the formula (7) will also satisfy the system, thus proving the 'if' part of theorem 3.2.

The proof is by induction on the order s of the determinants in Δ^{hi}_{IJ}. The case $s = 1$ is immediate since Δ^{hi}_{IJ} is of first order and, as the system (8) is of second order, it holds identically. The case $s = 2$ is exceptional because of the factor $(s-2)^{-1}$ that appears in the recurrence formula. Moreover, the proof of this case is particularly cumbersome since $\partial^2\Delta^{hi}_{IJ}/\partial y^b_{(uv)}\partial y^a_{(kl)}$ is a constant expressed as a function of the Kronecker deltas of the indices involved. Therefore, we are led to write a different proof for each type of equation (i.e. (I)–(III)) into which the system (8) decomposes. In the general case, the proof resorts to an analogous technique in order to state that the coefficients of the functions $Q^{ab,t}_{\alpha}$, defined below, vanish. These coefficients are expressed again in terms of certain Kronecker deltas of the involved indices.

We start with the following lemma.

Lemma 5.1. *With the same assumptions and notations as above we have*

$$\frac{\partial\Delta^{ii}_{IJ}}{\partial y^a_{\alpha}} = \frac{s}{s-1}\delta_{ia}\sum_{k,l=1}^{s}(-1)^{k+l}\delta_{\alpha,(i_k j_l)}\Delta^{ii}_{I-i_k,J-j_l} \tag{10}$$
$$(|\alpha| = 2;\ a, i = 1,\ldots,m;\ I, J \in \mathcal{I}_s,\ I \leqslant J,\ s \geqslant 2).$$

Proof. The formula in the statement follows, after a calculation, from the well known formula for the derivative of a functional determinant $\Delta = |C_1,\ldots,C_s|$, the columns of which are C_i; that is,

$$\Delta' = |C_1', C_2,\ldots,C_s| + |C_1, C_2',\ldots,C_s| + \cdots + |C_1,\ldots,C_{s-1},C_s'|.$$

□

Proposition 5.2. *The functions Δ^{hi}_{IJ} in section 3 satisfy the system (8).*

Proof. If $s = 1$ the result is obvious as the second derivatives of Δ^{hi}_{IJ} vanish identically in this case. This is because each Δ^{hi}_{IJ} is a linear function (see section 3–(ii)). If $s = 2$, then from (6) we have

$$\Delta^{hi}_{IJ} = \begin{vmatrix} y^h_{(i_1j_1)} & y^i_{(i_1j_2)} \\ y^h_{(i_2j_1)} & y^i_{(i_2j_2)} \end{vmatrix} + \begin{vmatrix} y^i_{(i_1j_1)} & y^h_{(i_1j_2)} \\ y^i_{(i_2j_1)} & y^h_{(i_2j_2)} \end{vmatrix} \qquad I = (i_1 < i_2) \qquad J = (j_1 < j_2)$$

and taking partial derivatives, from lemma 2.1 we obtain

$$\begin{aligned}\frac{\partial^2\Delta^{hi}_{IJ}}{\partial y^b_{(uv)}\partial y^a_{(kl)}} = {}& (\delta_{ha}\delta_{ib} + \delta_{ia}\delta_{hb}) \\ & \times[(\delta_{ki_1}\delta_{lj_1} + \delta_{kj_1}\delta_{li_1} - \delta_{kl}\delta_{i_1j_1}\delta_{kj_1})(\delta_{ui_2}\delta_{vj_2} + \delta_{uj_2}\delta_{vi_2} - \delta_{uv}\delta_{i_2j_2}\delta_{uj_2}) \\ & -(\delta_{ki_2}\delta_{lj_1} + \delta_{kj_1}\delta_{li_2} - \delta_{kl}\delta_{i_2j_1}\delta_{kj_1})(\delta_{ui_1}\delta_{vj_2} + \delta_{uj_2}\delta_{vi_1} - \delta_{uv}\delta_{i_1j_2}\delta_{uj_2}) \\ & +(\delta_{ki_2}\delta_{lj_2} + \delta_{kj_2}\delta_{li_2} - \delta_{kl}\delta_{i_2j_2}\delta_{kj_2})(\delta_{ui_1}\delta_{vj_1} + \delta_{uj_1}\delta_{vi_1} - \delta_{uv}\delta_{i_1j_1}\delta_{uj_1}) \\ & -(\delta_{ki_1}\delta_{lj_2} + \delta_{kj_2}\delta_{li_1} - \delta_{kl}\delta_{i_1j_2}\delta_{kj_2})(\delta_{ui_2}\delta_{vj_1} + \delta_{uj_1}\delta_{vi_2} - \delta_{uv}\delta_{i_2j_1}\delta_{uj_1})].\end{aligned} \tag{11}$$

According to equations (I)–(III) we have to distinguish three cases as follows.

(i) If $\alpha = (vvv)$, $v, l = 1, \ldots, n$, then from (11) we obtain

$$\frac{\partial^2 \Delta^{hi}_{IJ}}{\partial y^b_{(vv)} \partial y^a_{(vl)}} = (\delta_{ha}\delta_{ib} + \delta_{ia}\delta_{hb}) \times[\delta_{lj_2}(\delta_{vi_1}\delta_{vj_1}\delta_{i_2j_1} - \delta_{vi_2}\delta_{vj_1}\delta_{i_1j_1}) + \delta_{lj_1}(\delta_{vi_2}\delta_{vj_2}\delta_{i_1j_2} - \delta_{vi_1}\delta_{vj_2}\delta_{i_2j_2}) +\delta_{vj_2}\delta_{vj_1}(-\delta_{li_2}\delta_{i_1j_1} - \delta_{li_1}\delta_{i_2j_2} + \delta_{li_1}\delta_{i_2j_1} + \delta_{li_2}\delta_{i_1j_2}) +2\delta_{vj_1}\delta_{vj_2}\delta_{vl}(-\delta_{vi_2}\delta_{i_1j_1} - \delta_{vi_1}\delta_{i_2j_2} +\delta_{vi_1}\delta_{i_2j_1} + \delta_{vi_2}\delta_{i_1j_2} + \delta_{i_1j_1}\delta_{i_2j_2} - \delta_{i_2j_1}\delta_{i_1j_2})].$$

On the right-hand side of this formula the terms $\delta_{vi_1}\delta_{vj_1}\delta_{i_2j_1}$, $\delta_{vi_2}\delta_{vj_1}\delta_{i_1j_1}$, $\delta_{vi_2}\delta_{vj_2}\delta_{i_1j_2}$, $\delta_{vi_1}\delta_{vj_2}\delta_{i_2j_2}$ vanish as $v = i_1 = j_1$ implies $\delta_{i_2j_1} = 0$, and the same for the others. Similarly, the factors $\delta_{vj_2}\delta_{vj_1}$, $\delta_{vj_1}\delta_{vj_2}\delta_{vl}$ also vanish.

(ii.1) If $\alpha = (uvv)$, $u \neq v \neq l \neq u$, then from (11) we obtain

$$\frac{\partial^2 L}{\partial y^b_{(uv)} \partial y^a_{(vl)}} + \frac{\partial^2 L}{\partial y^b_{(vv)} \partial y^a_{(ul)}} = (\delta_{ha}\delta_{ib} + \delta_{ia}\delta_{hb})[\delta_{vj_2}(\delta_{vi_2} - \delta_{i_2j_2})(\delta_{ui_1}\delta_{lj_1} + \delta_{uj_1}\delta_{li_1}) +\delta_{vj_1}(\delta_{vi_1} - \delta_{i_1j_1})(\delta_{uj_2}\delta_{li_2} + \delta_{ui_2}\delta_{lj_2}) - \delta_{vj_2}(\delta_{vi_1} - \delta_{i_1j_2})(\delta_{ui_2}\delta_{lj_1} + \delta_{uj_1}\delta_{li_2}) -\delta_{vj_1}(\delta_{vi_2} - \delta_{i_2j_1})(\delta_{uj_2}\delta_{li_1} + \delta_{ui_1}\delta_{lj_2})]$$

and the right-hand side vanishes for

$$\delta_{vj_2}(\delta_{vi_2} - \delta_{i_2j_2}) = \delta_{vj_1}(\delta_{vi_1} - \delta_{i_1j_1}) = \delta_{vj_2}(\delta_{vi_1} - \delta_{i_1j_2}) = \delta_{vj_1}(\delta_{vi_2} - \delta_{i_2j_1}) = 0$$

as $i_1 < j_2$.

(ii.2) If $\alpha = (uvv)$, $l = u$, $v < u$, then from (11) we obtain

$$\frac{\partial^2 \Delta^{hi}_{IJ}}{\partial y^b_{(uv)} \partial y^a_{(uv)}} + 2\frac{\partial^2 \Delta^{hi}_{IJ}}{\partial y^b_{(vv)} \partial y^a_{(uu)}} = (4\delta_{ui_2}\delta_{uj_1} - 2\delta_{i_2j_1}\delta_{uj_1})\delta_{i_1j_2}\delta_{vj_2} +(4\delta_{i_2j_1}\delta_{uj_1} - 6\delta_{ui_2}\delta_{uj_1})\delta_{vi_1}\delta_{vj_2} + (4\delta_{i_2j_1} - 6\delta_{vi_2})\delta_{ui_1}\delta_{uj_2}\delta_{vj_1} +(4\delta_{vi_2} - 2\delta_{i_2j_1})\delta_{i_1j_2}\delta_{uj_2}\delta_{vj_1} +((6\delta_{ui_2} - 4\delta_{i_2j_2})\delta_{vi_1} + (2\delta_{i_2j_2} - 4\delta_{ui_2})\delta_{i_1j_1})\delta_{vj_1}\delta_{uj_2} +((6\delta_{vi_2} - 4\delta_{i_2j_2})\delta_{ui_1} + (2\delta_{i_2j_2} - 4\delta_{vi_2})\delta_{i_1j_1})\delta_{vj_2}\delta_{uj_1}.$$

On the right-hand side the first four terms vanish as $i_1 < j_2$. If $v = j_1$, $u = j_2$ the sixth term vanishes and so does the fifth, which is easily checked. Similarly, when $v = j_2$, $u = j_1$.

(iii) If $\alpha = (uvk)$, $u < v < k$, $l \notin \{u, v, k\}$, then, as a simple calculation using Maple V shows, from the formula (11) we conclude that the sum

$$\frac{\partial^2 \Delta^{hi}_{IJ}}{\partial y^b_{(uv)} \partial y^a_{(kl)}} + \frac{\partial^2 \Delta^{hi}_{IJ}}{\partial y^b_{(kv)} \partial y^a_{(ul)}} + \frac{\partial^2 \Delta^{hi}_{IJ}}{\partial y^b_{(ku)} \partial y^a_{(vl)}}$$

vanishes identically.

Now we proceed by induction on s assuming $s \geqslant 3$. Expanding the hth determinant in Δ^{hi}_{IJ} according to its hth column for $h = 1, \ldots, s$, we first obtain

$$\Delta^{hi}_{IJ} = \frac{1}{s-1}\sum_{k,l=1}^{s}(-1)^{k+l} y^h_{(i_kj_l)}\Delta^{ii}_{I-i_k,J-j_l} \tag{12}$$

and taking derivatives we have

$$\frac{\partial \Delta^{hi}_{IJ}}{\partial y^b_{(rt)}} = \frac{1}{s-1}\sum_{k,l=1}^{s}(-1)^{k+l}\left\{\delta_{hb}\delta_{(rt),(i_kj_l)}\Delta^{ii}_{I-i_k,J-j_l} + y^h_{(i_kj_l)}\frac{\partial \Delta^{ii}_{I-i_k,J-j_l}}{\partial y^b_{(rt)}}\right\}. \tag{13}$$

Again taking derivatives, and summing up,

$$(s-1)\sum_{r=1}^{n}\frac{1}{2-\delta_{rt}}\frac{\partial^2\Delta^{hi}_{IJ}}{\partial y^a_{\alpha-(r)}\partial y^b_{(rt)}} = Q^{ab,t}_{\alpha} + \sum_{k,l=1}^{s}(-1)^{k+l}y^h_{(i_k j_l)}\sum_{r=1}^{n}\frac{1}{2-\delta_{rt}}\frac{\partial^2\Delta^{ii}_{I-i_k,J-j_l}}{\partial y^a_{\alpha-(r)}\partial y^b_{(rt)}} \qquad (14)$$

where

$$Q^{ab,t}_{\alpha} = \sum_{k,l=1}^{s}\sum_{r=1}^{n}(-1)^{k+l}\left\{\delta_{hb}\frac{\delta_{(rt),(i_k j_l)}}{2-\delta_{rt}}\frac{\partial\Delta^{ii}_{I-i_k,J-j_l}}{\partial y^a_{\alpha-(r)}} + \delta_{ha}\frac{\delta_{\alpha-(r),(i_k j_l)}}{2-\delta_{rt}}\frac{\partial\Delta^{ii}_{I-i_k,J-j_l}}{\partial y^b_{(rt)}}\right\}.$$

By the induction hypothesis the second term on the right-hand side of (14) vanishes. Hence we are led to prove $Q^{ab,t}_{\alpha} = 0$, for every multi-index $\alpha \in \mathbb{N}^n$, $|\alpha| = 3$ and all $a, b = 1, \ldots, m$, $t = 1, \ldots, n$. By applying lemma 5.1 to the derivatives of $\Delta^{ii}_{I-i_k,J-j_l}$ in the formula above we obtain

$$Q^{ab,t}_{\alpha} = \frac{s-1}{s-2}(\delta_{hb}\delta_{ia} + \delta_{ha}\delta_{ib})\sum_{\substack{1\leqslant k'<k\leqslant s\\1\leqslant l'<l\leqslant s}}(-1)^{k+k'+l+l'}C^{kk',ll',t}_{\alpha}\Delta^{ii}_{I_{(kk')},J_{(ll')}}$$

with $\Delta^{ii}_{I_{(kk')},J_{(ll')}} = \Delta^{ii}_{I-\{i_k,i_{k'}\},J-\{j_l,j_{l'}\}}$ and

$$C^{kk',ll',t}_{\alpha} = \sum_{r=1}^{n}\frac{1}{2-\delta_{rt}}(\delta_{(rt),(i_k j_l)}\delta_{\alpha-(r),(i_{k'} j_{l'})} - \delta_{(rt),(i_k j_{l'})}\delta_{\alpha-(r),(i_{k'} j_l)}$$
$$-\delta_{(rt),(i_{k'} j_l)}\delta_{\alpha-(r),(i_k j_{l'})} + \delta_{(rt),(i_{k'} j_{l'})}\delta_{\alpha-(r),(i_k j_l)}).$$

As the functions $\Delta^{ii}_{I_{(kk')},J_{(ll')}}$ in the rank $1 \leqslant k' < k \leqslant s$, $1 \leqslant l' < l \leqslant s$ are linearly independent, $Q^{ab,t}_{\alpha} = 0$ if and only if $C^{kk',ll',t}_{\alpha} = 0$. Similar to the case $s = 2$ we have to distinguish three cases as follows.

(i) If $\alpha = (vvv)$, then a simple calculation by using lemma 2.1 yields

$$(2-\delta_{vt})C^{kk',ll',t}_{(vvv)} = \delta_{vj_l}\delta_{tj_{l'}}(\delta_{vi_k}\delta_{i_{k'}j_l} - \delta_{vi_{k'}}\delta_{i_k j_l}) + \delta_{vj_{l'}}\delta_{tj_l}(\delta_{vi_{k'}}\delta_{i_k j_{l'}} - \delta_{vi_k}\delta_{i_{k'}j_{l'}})$$
$$+\delta_{vj_l}\delta_{vj_{l'}}[(\delta_{ti_k} + 2\delta_{vt}\delta_{vi_k})(\delta_{i_{k'}j_l} - \delta_{i_{k'}j_{l'}})$$
$$+(\delta_{ti_{k'}} + 2\delta_{vt}\delta_{vi_{k'}})(\delta_{i_k j_{l'}} - \delta_{i_k j_l}) + 2\delta_{vt}(\delta_{i_k j_l}\delta_{i_{k'}j_{l'}} - \delta_{i_k j_{l'}}\delta_{i_{k'}j_l})] = 0$$

for $\delta_{vj_l}\delta_{vj_{l'}} = 0$, $\delta_{vi_k}\delta_{i_{k'}j_l} - \delta_{vi_{k'}}\delta_{i_k j_l}$ vanishes when $v = j_l$ and, analogously, $\delta_{vi_{k'}}\delta_{i_k j_{l'}} - \delta_{vi_k}\delta_{i_{k'}j_{l'}}$ also vanishes when $v = j_{l'}$.

(ii) Assume $\alpha = (uvv)$, with $u \neq v$. Then we have to consider three subcases as follows.

(ii.1) If $t \neq u$, $t \neq v$, then

$$2C^{kk',ll',t}_{(uvv)} = (\delta_{ui_k}\delta_{tj_l} + \delta_{uj_l}\delta_{ti_k})\delta_{vj_{l'}}(\delta_{vi_{k'}} - \delta_{i_{k'}j_{l'}})$$
$$+(\delta_{ui_k}\delta_{tj_{l'}} + \delta_{uj_{l'}}\delta_{ti_k})\delta_{vj_l}(\delta_{i_{k'}j_l} - \delta_{vi_{k'}})$$
$$+(\delta_{ui_{k'}}\delta_{tj_l} + \delta_{uj_l}\delta_{ti_{k'}})\delta_{vj_{l'}}(\delta_{i_k j_{l'}} - \delta_{vi_k})$$
$$+(\delta_{ui_{k'}}\delta_{tj_{l'}} + \delta_{uj_{l'}}\delta_{ti_{k'}})\delta_{vj_l}(\delta_{vi_k} - \delta_{i_k j_l})$$

and the right-hand side vanishes as

$$\delta_{vj_{l'}}(\delta_{vi_{k'}} - \delta_{i_{k'}j_{l'}}) = \delta_{vj_l}(\delta_{i_{k'}j_l} - \delta_{vi_{k'}})$$
$$= \delta_{vj_{l'}}(\delta_{i_k j_{l'}} - \delta_{vi_k}) = \delta_{vj_l}(\delta_{vi_k} - \delta_{i_k j_l}) = 0.$$

(ii.2) If $t = u$, then $C^{kk',ll',u}_{(uvv)} = s_1 + s_2 + s_3 + s_4$, where

$$s_1 = \delta_{uj_l}\delta_{vj_{l'}}(3\delta_{ui_k}\delta_{vi_{k'}} + \delta_{i_k j_l}\delta_{i_{k'}j_{l'}} - 2\delta_{i_k j_l}\delta_{vi_{k'}} - 2\delta_{ui_k}\delta_{i_{k'}j_{l'}})$$
$$s_2 = \delta_{uj_l}\delta_{vj_{l'}}(3\delta_{ui_{k'}}\delta_{vi_k} + \delta_{i_{k'}j_l}\delta_{i_k j_{l'}} - 2\delta_{i_{k'}j_l}\delta_{vi_k} - 2\delta_{ui_{k'}}\delta_{i_k j_{l'}})$$
$$s_3 = \delta_{uj_{l'}}\delta_{vj_l}(3\delta_{ui_k}\delta_{vi_{k'}} + \delta_{i_{k'}j_l}\delta_{i_k j_{l'}} - 2\delta_{i_k j_{l'}}\delta_{vi_{k'}} - 2\delta_{ui_k}\delta_{i_{k'}j_l})$$
$$s_4 = \delta_{uj_{l'}}\delta_{vj_l}(3\delta_{ui_{k'}}\delta_{vi_k} + \delta_{i_k j_l}\delta_{i_{k'}j_{l'}} - 2\delta_{i_{k'}j_{l'}}\delta_{vi_k} - 2\delta_{ui_{k'}}\delta_{i_k j_l})$$

and it is easily checked that the right-hand side of the equations above vanish identically. Hence $C^{kk',ll',u}_{(uvv)} = 0$.

(ii.3) If $t = v$, then

$$\tfrac{2}{3}C^{kk',ll',v}_{(uvv)} = \delta_{vj_l}\delta_{vj_{l'}}[(\delta_{i_k j_{l'}} - \delta_{i_k j_l})\delta_{ui_{k'}} + (\delta_{i_{k'} j_l} - \delta_{i_{k'} j_{l'}})\delta_{ui_k}] + \delta_{uj_l}\delta_{vj_{l'}}(\delta_{i_k j_{l'}}\delta_{vi_{k'}} - \delta_{i_{k'} j_{l'}}\delta_{vi_k}) + \delta_{uj_{l'}}\delta_{vj_l}(\delta_{i_{k'} j_l}\delta_{vi_k} - \delta_{i_k j_l}\delta_{vi_{k'}})$$

and $C^{kk',ll',v}_{(uvv)}$ vanishes as

$$\delta_{vj_l}\delta_{vj_{l'}} = \delta_{vj_{l'}}(\delta_{i_k j_{l'}}\delta_{vi_{k'}} - \delta_{i_{k'} j_{l'}}\delta_{vi_k}) = \delta_{vj_l}(\delta_{i_{k'} j_l}\delta_{vi_k} - \delta_{i_k j_l}\delta_{vi_{k'}}) = 0.$$

(iii) If $\alpha = (uvw)$ with $u \neq v \neq w \neq u$, then we have to consider two subcases as follows.

(iii.1) If $t \notin \{u, v, w\}$, then $C^{kk',ll',v}_{(uvw)} = 0$ as a simple computation shows.

(iii.2) If $t \in \{u, v, w\}$, say $t = u$, then $C^{kk',ll',u}_{(uvw)} = s_1 + s_2 + s_3 + s_4$, where

$$s_1 = (\delta_{vi_{k'}}\delta_{wj_{l'}} + \delta_{vj_{l'}}\delta_{wi_{k'}})\delta_{uj_l}(\delta_{ui_k} - \delta_{i_k j_l})$$
$$s_2 = -(\delta_{vi_k}\delta_{wj_{l'}} + \delta_{vj_{l'}}\delta_{wi_k})\delta_{uj_l}(\delta_{ui_{k'}} - \delta_{i_{k'} j_l})$$
$$s_3 = -(\delta_{vi_{k'}}\delta_{wj_l} + \delta_{vj_l}\delta_{wi_{k'}})\delta_{uj_{l'}}(\delta_{ui_k} - \delta_{i_k j_{l'}})$$
$$s_4 = (\delta_{vi_k}\delta_{wj_l} + \delta_{vj_l}\delta_{wi_k})\delta_{uj_{l'}}(\delta_{ui_{k'}} - \delta_{i_{k'} j_{l'}})$$

and every summand vanishes as

$$\delta_{uj_l}(\delta_{ui_k} - \delta_{i_k j_l}) = \delta_{uj_l}(\delta_{ui_{k'}} - \delta_{i_{k'} j_l}) = \delta_{uj_{l'}}(\delta_{ui_k} - \delta_{i_k j_{l'}}) = \delta_{uj_{l'}}(\delta_{ui_{k'}} - \delta_{i_{k'} j_{l'}}) = 0.$$

For $t = v$ or w the proof runs analogously.

This completes the proof of proposition 5.2. □

5.2. The 'only if' part

Next, we prove the 'only if' part of theorem 3.2.

As the determinants Δ^{hi}_{IJ} are polynomials of degree $\leqslant n$, the first step in the proof consists of proving that an arbitrary solution L to the system (8) is also a polynomial of degree $\leqslant n$ (see proposition 5.3). Moreover, as the system (8) is homogeneous, it suffices to state the proof for the homogeneous solutions to (8) of degree say s, with respect to the second-order variables $y^i_{(jk)}$. Starting from this point the proof is by recurrence on s. For $s = 0$ or $s = 1$ the result is trivial. As in the 'if' part, the case $s = 2$ is exceptional because of the factor $(s-2)^{-1}$ appearing in the recurrence formula. To prove this case we first need to establish certain symmetries of the second derivatives of L (see lemma 5.4 below).

For the general case, the basic idea is to use the fact that if L is a solution to (8), then so is $\partial L/\partial y^\alpha_{(\beta\gamma)}$. Then, we apply the recurrence hypothesis to this derivative (see the formula (22)). Since L is homogeneous of degree $s+1$, from Euler's theorem we deduce the formula (23). The rest of the proof—in essence a long computation—reduces to state that certain symmetries of the functions $f^{(bc),IJ}_{a,hi}$ hold true allowing us to factor out the determinants $\Delta^{hi}_{I'J'}$ of order $s+1$ in the expression of L. This finishes the proof.

Proposition 5.3. *If a function $L \in C^\infty(J^2)$ satisfies the system (8), then L is a polynomial function of degree $\leqslant n$ in the variables y^i_α, $1 \leqslant i \leqslant m$, $|\alpha| = 2$, with coefficients in $C^\infty(J^1)$.*

Proof. We only need to prove that for all indices $i_0, i_1, \ldots, i_n = 1, \ldots, m, a_0 \leqslant a_1 \leqslant \cdots \leqslant a_n$, $1 \leqslant a_i \leqslant b_i \leqslant n$, $0 \leqslant i \leqslant n$, we have

$$\frac{\partial^{n+1}L}{\partial y^{i_0}_{(a_0b_0)}\partial y^{i_1}_{(a_1b_1)}\cdots\partial y^{i_n}_{(a_nb_n)}} = 0. \tag{15}$$

In fact, taking derivatives with respect to $\partial/\partial y^j_{(ad)}$ on the equation (II.1), we obtain

$$\frac{\partial^3 L}{\partial y^h_{(ab)}\partial y^i_{(ac)}\partial y^j_{(ad)}} + \frac{\partial^3 L}{\partial y^h_{(aa)}\partial y^j_{(ad)}\partial y^i_{(bc)}} = 0 \tag{16}$$

for all indices $h, i, j = 1, \ldots, m$, $a \neq b \neq c \neq a$, $a, b, c, d = 1, \ldots, n$. From equation (I) we conclude that the second summand on the left-hand side in (16) vanishes. Similarly, again taking derivatives with respect to $\partial/\partial y^j_{(ad)}$ on the equation (II.2) yields

$$\frac{\partial^3 L}{\partial y^h_{(ab)}\partial y^i_{(ab)}\partial y^j_{(ad)}} + 2\frac{\partial^3 L}{\partial y^h_{(aa)}\partial y^j_{(ad)}\partial y^i_{(bb)}} = 0$$

for all indices $h, i, j = 1, \ldots, m$, $a \neq b$, $a, b, d = 1, \ldots, n$, and proceeding as above we finally obtain

$$\frac{\partial^3 L}{\partial y^h_{(ab)}\partial y^i_{(ac)}\partial y^j_{(ad)}} = 0 \tag{17}$$

for all indices $h, i, j = 1, \ldots, m$, $a \neq b$, $a \neq c$, $a, b, c, d = 1, \ldots, n$.

Moreover, we remark that every sequence of bi-indices $(a_0b_0), \ldots, (a_nb_n)$ such that $a_0 \leqslant a_1 \leqslant \cdots \leqslant a_n$, $1 \leqslant a_i \leqslant b_i \leqslant n$, $0 \leqslant i \leqslant n$, satisfies either

(i) there exist two indices $0 \leqslant i < j \leqslant n$, such that $a_i = b_i = a_j$ or $a_i = b_i = b_j$, or
(ii) there exist three indices $0 \leqslant i < j < k \leqslant n$, such that $a_i = a_j = a_k$,

as it is easily checked that the only sequences of maximal length satisfying neither (i) nor (ii) are $(12), (13), \ldots, (i, i+2), \ldots, (n-1, n)$ and $(11), (22), \ldots, (nn)$, both of length n. Hence from (17) and taking into account (i) and (ii), we conclude that (15) holds, thus finishing the proof. □

According to proposition 5.3 every solution to the system (8) can be written as $L = \sum_{s=0}^n L_s$, with

$$\begin{array}{lll} rL_s = f^{\alpha_1\ldots\alpha_s}_{i_1\cdots i_s} y^{i_1}_{\alpha_1}\cdots y^{i_s}_{\alpha_s} & & f^{\alpha_1\cdots\alpha_s}_{i_1\cdots i_s} \in C^\infty(J^1) \\ |\alpha_h| = 2 & 1 \leqslant h \leqslant s & i_1, \ldots, i_s = 1, \ldots, m. \end{array} \tag{18}$$

As equations (8) are homogeneous, L is a solution if and only every L_s is a solution to this system. Hence from now on we assume that L itself is homogeneous; i.e., there exists an integer $0 \leqslant s \leqslant n$, such that

$$\begin{array}{lll} rL = f^{a_1b_1\ldots a_sb_s}_{i_1\cdots i_s} y^{i_1}_{(a_1b_1)}\cdots y^{i_s}_{(a_sb_s)} & & f^{a_1b_1\cdots a_sb_s}_{i_1\cdots i_s} \in C^\infty(J^1) \\ a_h \leqslant b_h & 1 \leqslant h \leqslant s & a_1 \leqslant \cdots \leqslant a_s. \end{array} \tag{19}$$

For $s = 0$ or 1, the result is trivial (see (i), (ii) in section 3). For $s = 2$ we proceed directly. First, we state a general result in the following:

Lemma 5.4. *Let L be a solution to the system (8). For arbitrary indices $h, i = 1, \ldots, m$; $a, b, c, d = 1, \ldots, n$, $a \leqslant b$, $c \leqslant d$, we have*

$$\frac{\partial^2 L}{\partial y^h_{(ab)}\partial y^i_{(cd)}} = \frac{\partial^2 L}{\partial y^i_{(ab)}\partial y^h_{(cd)}}.$$

Proof. If $h = i$, the result is trivial, so suppose $h \neq i$. First, we assume a, b, c, d are pairwise different. Then, from equation (III.1) we obtain

$$\frac{\partial^2 L}{\partial y^h_{(bc)}\partial y^i_{(ad)}} + \frac{\partial^2 L}{\partial y^h_{(ac)}\partial y^i_{(bd)}} + \frac{\partial^2 L}{\partial y^h_{(ab)}\partial y^i_{(cd)}} = 0.$$

Permuting c and d,

$$\frac{\partial^2 L}{\partial y^h_{(bd)}\partial y^i_{(ac)}} + \frac{\partial^2 L}{\partial y^h_{(ad)}\partial y^i_{(bc)}} + \frac{\partial^2 L}{\partial y^h_{(ab)}\partial y^i_{(cd)}} = 0$$

and permuting h and i in the former equation,

$$\frac{\partial^2 L}{\partial y^h_{(ad)}\partial y^i_{(bc)}} + \frac{\partial^2 L}{\partial y^h_{(bd)}\partial y^i_{(ac)}} + \frac{\partial^2 L}{\partial y^h_{(cd)}\partial y^i_{(ab)}} = 0.$$

Subtracting the last two equations we reach the required conclusion. Next, we assume $d = a$ (the cases $d = b$, $d = c$ are the same). Then, from equation (II.1.(a)) we obtain

$$\frac{\partial^2 L}{\partial y^h_{(ab)}\partial y^i_{(ac)}} + \frac{\partial^2 L}{\partial y^h_{(aa)}\partial y^i_{(bc)}} = 0$$

and permuting b and c, and comparing both equations, we reach the required conclusion. If $a = b$ and $c = d$ but $a \neq c$, then from equation (II.2), permuting a and c we have

$$\frac{\partial^2 L}{\partial y^h_{(ac)}\partial y^i_{(ac)}} + 2\frac{\partial^2 L}{\partial y^h_{(aa)}\partial y^i_{(cc)}} = 0 \qquad \frac{\partial^2 L}{\partial y^h_{(ac)}\partial y^i_{(ac)}} + 2\frac{\partial^2 L}{\partial y^h_{(cc)}\partial y^i_{(aa)}} = 0$$

and comparing both equations we have our conclusion. Finally, the cases in which three or four of the indices a, b, c, d coincide are trivial because of equation (I). □

Next we continue with the proof of the case $s = 2$. If L is a second-degree homogeneous solution to equations (8) then the Taylor expansion yields

$$L = \frac{1}{2}\sum_{h,i}\sum_{a\leqslant b}\sum_{c\leqslant d}\frac{\partial^2 L}{\partial y^h_{(ab)}\partial y^i_{(cd)}} y^h_{(ab)} y^i_{(cd)}. \tag{20}$$

We assume $a < b < c$, $d \neq a, b, c$, $h \neq i$. Then the terms involving the indices h, i, and a, b, c, d are the following:

$$\frac{1}{2}\left(\frac{\partial^2 L}{\partial y^h_{(ab)}\partial y^i_{(cd)}} y^h_{(ab)} y^i_{(cd)} + \frac{\partial^2 L}{\partial y^h_{(ac)}\partial y^i_{(bd)}} y^h_{(ac)} y^i_{(bd)} + \frac{\partial^2 L}{\partial y^h_{(ad)}\partial y^i_{(bc)}} y^h_{(ad)} y^i_{(bc)}\right.$$
$$\left. + \frac{\partial^2 L}{\partial y^h_{(cd)}\partial y^i_{(ab)}} y^h_{(cd)} y^i_{(ab)} + \frac{\partial^2 L}{\partial y^h_{(bd)}\partial y^i_{(ac)}} y^h_{(bd)} y^i_{(ac)} + \frac{\partial^2 L}{\partial y^h_{(bc)}\partial y^i_{(ad)}} y^h_{(bc)} y^i_{(ad)}\right).$$

Using equations (III.1) and taking into account lemma 5.4, the sum above is readily seen to be equal to

$$\frac{1}{2}\left(\frac{\partial^2 L}{\partial y^h_{(ab)}\partial y^i_{(cd)}} (y^h_{(ab)} y^i_{(cd)} + y^h_{(cd)} y^i_{(ab)} - y^h_{(bc)} y^i_{(ad)} - y^h_{(ad)} y^i_{(bc)})\right.$$
$$\left. + \frac{\partial^2 L}{\partial y^h_{(ac)}\partial y^i_{(bd)}} (y^h_{(ac)} y^i_{(bd)} + y^h_{(bd)} y^i_{(ac)} - y^h_{(bc)} y^i_{(ad)} - y^h_{(ad)} y^i_{(bc)})\right)$$
$$= \frac{1}{2}\left(\frac{\partial^2 L}{\partial y^h_{(ab)}\partial y^i_{(cd)}} \Delta^{hi}_{(ac)(bd)} + \frac{\partial^2 L}{\partial y^h_{(ac)}\partial y^i_{(bd)}} \Delta^{hi}_{(ab)(cd)}\right). \tag{21}$$

The case where $a = b = c$ is trivial by virtue of equation (I). When $a = b$, (or $c = d$), the sum above is now

$$\frac{1}{2}\frac{\partial^2 L}{\partial y^h_{(ac)}\partial y^i_{(ad)}} \Delta^{hi}_{(ac)(ad)}.$$

Last, if $a = c$, $b = d$, this yields

$$\frac{1}{2}\frac{\partial^2 L}{\partial y^h_{(aa)}\partial y^i_{(bb)}}\Delta^{hi}_{(ab)(ab)}.$$

But these two cases may be seen as mere specializations of the general formula (21) with suitable values of a, b, c and d. So, for $s = 2$ and remembering that L is a second-degree polynomial, it becomes apparent that L is always a linear combination of at most two 'Deltas'.

Next, we assume $2 \leqslant s < n$ and that theorem 3.2 holds for every homogeneous function L of degree s. Let us consider a homogeneous solution L of degree $s+1$ to equations (I)–(III) and let us fix three indices $1 \leqslant \alpha \leqslant m$, $1 \leqslant \beta \leqslant \gamma \leqslant n$. As equations (I)–(III) are linear, homogeneous, and with constant coefficients, it is clear that $\partial L/\partial y^\alpha_{(\beta\gamma)}$ is also a solution to this system. Hence, by virtue of the induction hypothesis, we have

$$\frac{\partial L}{\partial y^\alpha_{(\beta\gamma)}} = f^{(\beta\gamma),IJ}_{\alpha,hi}\Delta^{hi}_{IJ} \qquad f^{\alpha,IJ}_{(\beta\gamma),hi} \in C^\infty(J^1(N,M)). \tag{22}$$

In the previous formula we have applied Einstein's convention for repeated indices, and we will do so from now on, since otherwise the formulas become too long.

As L is homogeneous of degree $s+1$, from Euler's theorem we obtain

$$(s+1)L = y^a_{(bc)}\frac{\partial L}{\partial y^a_{(bc)}} = f^{(bc),IJ}_{a,hi}\, y^a_{(bc)}\Delta^{hi}_{IJ} \tag{23}$$

and taking derivatives with respect to $y^\alpha_{(\beta\gamma)}$ on both sides of (23) from equation (22) we have

$$(s+1)f^{(\beta\gamma),IJ}_{\alpha,hi}\Delta^{hi}_{IJ} = (s+1)\frac{\partial L}{\partial y^\alpha_{(\beta\gamma)}} = f^{(\beta\gamma),IJ}_{\alpha,hi}\Delta^{hi}_{IJ} + f^{(bc),IJ}_{a,hi}\, y^a_{(bc)}\frac{\partial \Delta^{hi}_{IJ}}{\partial y^\alpha_{(\beta\gamma)}}.$$

Hence

$$s f^{(\beta\gamma),IJ}_{\alpha,hi}\Delta^{hi}_{IJ} = f^{(bc),IJ}_{a,hi}\, y^a_{(bc)}\frac{\partial \Delta^{hi}_{IJ}}{\partial y^\alpha_{(\beta\gamma)}}. \tag{24}$$

Before going on, let us introduce a notation. We set: if $I = (i_1, \ldots, i_s)$ and $\beta \in I$, then $u = \nu_I(\beta)$ means $\beta = i_u$. The following is quickly checked:

$$\nu_{I-i_k}(\beta) = \begin{cases} \nu_I(\beta) - 1 & \text{if} \quad k < \nu_I(\beta) \\ \text{not defined} & \text{if} \quad k = \nu_I(\beta) \\ \nu_I(\beta) & \text{if} \quad k > \nu_I(\beta). \end{cases} \tag{25}$$

As a simple computation shows, by using the very definition of Δ^{ii}_{IJ} we have

$$\frac{\partial \Delta^{ii}_{IJ}}{\partial y^\alpha_{(\beta\gamma)}} = \frac{s}{s-1}\delta_{\alpha i}\left((-1)^{\nu_I(\beta)+\nu_J(\gamma)}\Delta^{ii}_{I-\beta,J-\gamma} + (-1)^{\nu_J(\beta)+\nu_I(\gamma)}\Delta^{ii}_{I-\gamma,J-\beta}\right). \tag{26}$$

From formula (13) and using (26)

$$\begin{aligned}\frac{\partial \Delta^{hi}_{IJ}}{\partial y^\alpha_{(\beta\gamma)}} = \frac{1}{s-1}\sum_{k,l=1}^{s}(-1)^{k+l}\Bigg\{&\delta_{h\alpha}\delta_{(\beta\gamma),(i_k j_l)}\Delta^{ii}_{I-i_k,J-j_l} \\ &+\frac{s-1}{s-2}\delta_{\alpha i}\, y^h_{(i_k j_l)}\big[(-1)^{\nu_{I-i_k}(\beta)+\nu_{J-j_l}(\gamma)}\Delta^{ii}_{I-\beta-i_k,J-\gamma-j_l} \\ &+(-1)^{\nu_{J-j_l}(\beta)+\nu_{I-i_k}(\gamma)}\Delta^{ii}_{I-\gamma-i_k,J-\beta-j_l}\big]\Bigg\}.\end{aligned} \tag{27}$$

Hence equation (24) reads

$$sf^{(\beta\gamma),IJ}_{\alpha,hi}\Delta^{hi}_{IJ} = f^{(bc),IJ}_{a,hi}y^a_{(bc)}\left(\frac{1}{s-1}\sum_{k,l=1}^{s}(-1)^{k+l}\left\{\delta_{h\alpha}\delta_{(\beta\gamma),(i_kj_l)}\Delta^{ii}_{I-i_k,J-j_l} + \frac{s-1}{s-2}\delta_{\alpha i}y^h_{(i_kj_l)}[(-1)^{\nu_{I-i_k}(\beta)+\nu_{J-j_l}(\gamma)}\Delta^{ii}_{I-\beta-i_k,J-\gamma-j_l} + (-1)^{\nu_{J-j_l}(\beta)+\nu_{I-i_k}(\gamma)}\Delta^{ii}_{I-\gamma-i_k,J-\beta-j_l}]\right\}\right) \tag{28}$$

and using formula (25) the preceding equation becomes

$$\begin{aligned}
sf^{(\beta\gamma),IJ}_{\alpha,hi}\Delta^{hi}_{IJ} &= \frac{1}{s-1}f^{(bc),IJ}_{a,hi}y^a_{(bc)}\sum_{k,l=1}^{s}(-1)^{k+l}\delta_{h\alpha}\delta_{(\beta\gamma),(i_kj_l)}\Delta^{ii}_{I-i_k,J-j_l}\\
&\quad+\frac{1}{s-2}f^{(bc),IJ}_{a,h\alpha}y^a_{(bc)}\left[\sum_{k,l=1}^{s+1}(-1)^{k+l}y^h_{(i_kj_l)}(-1)^{\nu_{I-i_k}(\beta)+\nu_{J-j_l}(\gamma)}\right.\\
&\quad\left.\times\Delta^{\alpha\alpha}_{I-\beta-i_k,J-\gamma-j_l} + (-1)^{\nu_{I-i_k}(\gamma)+\nu_{J-j_l}(\beta)}\Delta^{\alpha\alpha}_{I-\gamma-i_k,J-\beta-j_l}\right]\\
&= \frac{1}{s-1}f^{(bc),IJ}_{a,\alpha i}y^a_{(bc)}((-1)^{\nu_I(\beta)+\nu_J(\gamma)}\Delta^{ii}_{I-\beta,J-\gamma} + (-1)^{\nu_I(\gamma)+\nu_J(\beta)}\Delta^{ii}_{I-\gamma,J-\beta})\\
&\quad+\frac{(-1)^{\nu_I(\beta)+\nu_J(\gamma)}}{s-2}f^{(bc),IJ}_{a,h\alpha}y^a_{(bc)}\left(\sum_{\substack{1\leqslant k<\nu_I(\beta)\\1\leqslant l<\nu_J(\gamma)}}(-1)^{k+l}y^h_{(i_kj_l)}\Delta^{\alpha\alpha}_{I-\beta-i_k,J-\gamma-j_l}\right.\\
&\quad\left.+\sum_{\substack{s\geqslant k>\nu_I(\beta)\\s\geqslant l>\nu_J(\gamma)}}(-1)^{k+l}y^h_{(i_kj_l)}\Delta^{\alpha\alpha}_{I-\beta-i_k,J-\gamma-j_l}\right)\\
&\quad+\frac{(-1)^{\nu_I(\gamma)+\nu_J(\beta)}}{s-2}f^{(bc),IJ}_{a,h\alpha}y^a_{(bc)}\left(\sum_{\substack{1\leqslant k<\nu_I(\gamma)\\1\leqslant l<\nu_J(\beta)}}(-1)^{k+l}y^h_{(i_kj_l)}\Delta^{\alpha\alpha}_{I-\gamma-i_k,J-\beta-j_l}\right.\\
&\quad\left.+\sum_{s\geqslant k>\nu_I(\gamma),s\geqslant l>\nu_J(\beta)}(-1)^{k+l}y^h_{(i_kj_l)}\Delta^{\alpha\alpha}_{I-\gamma-i_k,J-\beta-j_l}\right)\\
&\quad-\frac{(-1)^{\nu_I(\beta)+\nu_J(\gamma)}}{s-2}f^{(bc),IJ}_{a,h\alpha}y^a_{(bc)}\left(\sum_{\substack{1\leqslant k<\nu_I(\beta)\\s\geqslant l>\nu_J(\gamma)}}(-1)^{k+l}y^h_{(i_kj_l)}\Delta^{\alpha\alpha}_{I-\beta-i_k,J-\gamma-j_l}\right.\\
&\quad\left.+\sum_{\substack{s\geqslant k>\nu_I(\beta)\\1\leqslant l<\nu_J(\gamma)}}(-1)^{k+l}y^h_{(i_kj_l)}\Delta^{\alpha\alpha}_{I-\beta-i_k,J-\gamma-j_l}\right)\\
&\quad-\frac{(-1)^{\nu_I(\gamma)+\nu_J(\beta)}}{s-2}f^{(bc),IJ}_{a,h\alpha}y^a_{(bc)}\left(\sum_{\substack{1\leqslant k<\nu_I(\gamma)\\s\geqslant l>\nu_J(\beta)}}(-1)^{k+l}y^h_{(i_kj_l)}\Delta^{\alpha\alpha}_{I-\gamma-i_k,J-\beta-j_l}\right.\\
&\quad\left.+\sum_{\substack{s\geqslant k>\nu_I(\gamma)\\1\leqslant l<\nu_J(\beta)}}(-1)^{k+l}y^h_{(i_kj_l)}\Delta^{\alpha\alpha}_{I-\gamma-i_k,J-\beta-j_l}\right).
\end{aligned} \tag{29}$$

In the previous equation, the second and fourth brackets match together yielding $\Delta^{h\alpha}_{I-\beta,J-\gamma}$ and so do the third and the fifth yielding $\Delta^{h\alpha}_{I-\gamma,J-\beta}$. Hence

$$sf^{(\beta\gamma),IJ}_{\alpha,hi}\Delta^{hi}_{IJ} = \frac{1}{s-1}f^{(bc),IJ}_{a,\alpha i}y^a_{(bc)}[(-1)^{\nu_I(\beta)+\nu_J(\gamma)}\Delta^{ii}_{I-\beta,J-\gamma} + (-1)^{\nu_I(\gamma)+\nu_J(\beta)}\Delta^{ii}_{I-\gamma,J-\beta}]$$

$$+f^{(bc),IJ}_{a,h\alpha}y^a_{(bc)}[(-1)^{\nu_I(\beta)+\nu_J(\gamma)}\Delta^{h\alpha}_{I-\beta,J-\gamma}+(-1)^{\nu_I(\gamma)+\nu_J(\beta)}\Delta^{h\alpha}_{I-\gamma,J-\beta}]. \tag{30}$$

If $s=2$, then equations (27)–(29) have no meaning as in this case the term $\partial\Delta^{ii}_{I-i_k,J-j_l}/\partial y^b_{(rt)}$ is a Kronecker delta and hence formula (26) need not be applied. So, in this case, we can directly skip to formula (30).

Moreover, from formula (6) we obtain

$$\Delta^{hi}_{IJ}=\sum_{r=1}^{s}\sum_{\pi}\operatorname{sign}(\pi)y^i_{(i_1j_{\pi(1)})}\dots y^h_{(i_rj_{\pi(r)})}\dots y^i_{(i_sj_{\pi(s)})}$$

where π runs for the permutations of the indices $1,\dots,s$. Comparing the coefficients of the terms $\operatorname{sign}(\pi)y^i_{(i_1j_{\pi(1)})}\dots y^h_{(i_rj_{\pi(r)})}\dots y^i_{(i_sj_{\pi(s)})}$ in the Deltas on both sides of (30) we obtain the following properties:

(p$_1$) $f^{(bc),IJ}_{a,hi}=0$, whenever $a\neq h\neq i\neq a$.
(p$_2$) $sf^{(bc),IJ}_{h,ii}=f^{(bc),IJ}_{i,hi}$.
(p$_3$) $f^{(bc),IJ}_{a,hi}=0$, if ($b\in I$ or $c\in J$) and ($c\in I$ or $b\in J$).
(p$_4$) $(-1)^{\nu_J(c)}f^{(bc),I,J-c}_{i,hi}=(-1)^{\nu_J(d)}f^{(bd),I,J-d}_{i,hi}$.

According to (p$_4$) there are functions f^{IJ}_{hi} such that

$$f^{(bc),I,J-c}_{i,hi}=(-1)^{\nu_J(c)}f^{b,IJ}_{hi}.$$

Hence from (p$_2$), we have

$$f^{(bc),IJ}_{h,ii}=(-1)^{\nu_J(c)}s^{-1}f^{b,IJ}_{hi}.$$

Substituting these expressions in (23) and taking into account the properties (p$_1$), (p$_3$) we obtain

$$\begin{aligned}(s+1)L&=f^{(bc),IJ}_{i,hi}y^i_{(bc)}\Delta^{hi}_{IJ}+f^{(bc),IJ}_{h,ii}y^h_{(bc)}\Delta^{ii}_{IJ}\\&=(-1)^{\nu_J(c)}f^{b,IJ}_{hi}(y^i_{(bc)}\Delta^{hi}_{IJ}+s^{-1}y^h_{(bc)}\Delta^{ii}_{IJ}).\end{aligned} \tag{31}$$

Moreover, once a row u has been fixed in $\Delta^{hi}_{I'J'}$, with $I',J'\in\mathcal{I}_{s+1}$, then the following expansion holds:

$$\Delta^{hi}_{I'J'}=\sum_{v=1}^{s+1}(-1)^{u+v}[y^i_{(i'_uj'_v)}\Delta^{hi}_{I'-i_u,J'-j'_v}+s^{-1}y^h_{(i'_uj'_v)}\Delta^{ii}_{I'-i_u,J'-j'_v}]. \tag{32}$$

By using (32) in (31) we finally obtain

$$\begin{aligned}(s+1)L&=(-1)^{\nu_I(b)}f^{b,IJ}_{hi}(-1)^{\nu_I(b)+\nu_J(c)}[y^i_{(bc)}\Delta^{hi}_{IJ}+s^{-1}y^h_{(bc)}\Delta^{ii}_{IJ}]\\&=\tilde f^{I'J'}_{hi}\Delta^{hi}_{I'J'}\end{aligned}$$

for certain functions $\tilde f^{I'J'}_{hi}$, thus reaching our conclusion.

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